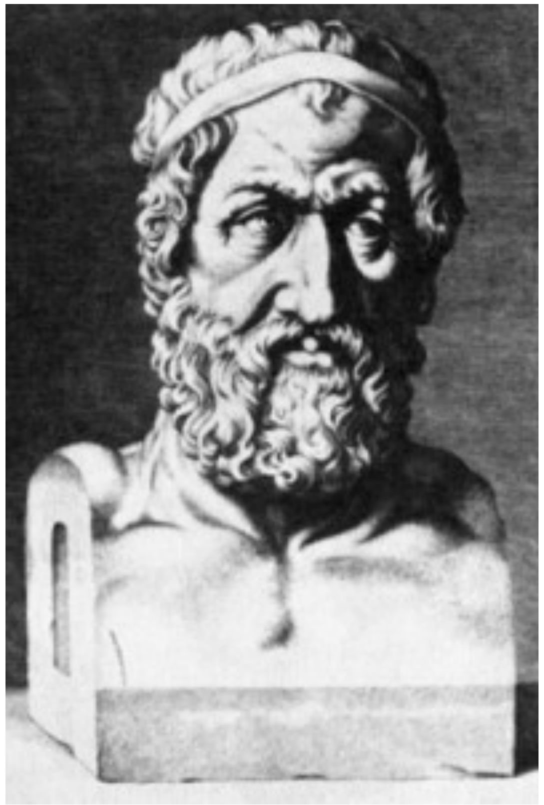
Zeno’s paradoxes

Our topic today will be a group of the oldest, and most historically important, paradoxes ever set forth: the paradoxes of motion credited to Zeno of Elea.



Zeno lived in the 5th century B.C. in a Greek colony in the southern portion of the Italian peninsula. Unfortunately, none of his writings survive. What we know of them is due to reports in the writings of other ancient philosophers, particularly Aristotle. Hence reconstructing his arguments is partly a matter of conjecture.

Zeno's paradoxes can be thought of as one of the earliest examples of a type of argument which has been quite common in the history of philosophy: an argument which, if successful, shows that some part of our ordinary picture of the world leads to contradiction. Zeno's idea was that a very basic part of our world-view — the view that things move — leads to contradiction.

You might wonder: how could anyone doubt that things move?

The idea of a thing moving is, to a first approximation, the idea of a certain physical thing - something which takes up space — occupying different bits of space at different times. One might think that nothing moves if one thinks that the physical world — the world of things which are extended in space — is illusory. This view is often called **idealism**.

On some reports, Zeno had upwards of 40 arguments against the reality of motion, most of which are unknown to us. We will be discussing four of his arguments:

The Racetrack   The Achilles   The Stadium   The Arrow

The Racetrack
The Achilles

The Stadium
The Arrow

These four paradoxes can be usefully separated into two groups.

To understand the reason for the grouping, we have to introduce the idea of a **continuous series**. For our purposes (though this is a simplification), a continuous series is one in which between every two members of the series, there is another member of the series.

Can you think of examples of continuous series (or continua) in this sense?

Consider the set of rational numbers. Can we ever come up with a pair of rational numbers — say, 1.23456 and 1.23457 — which are so close together that there is no rational number in between them?

Our question is: **are space and time continuous?** If they are, then between any two points in space there is a third. A consequence of space being continuous would be that, for any length, there is such a thing as half of that length. Applied to time, the idea would be that for any amount of time, there is such a thing as half that time, and that in between any two moments there is another.

If space and time are **not** continuous, then we say that they are **discrete**. If space is discrete, then there are lengths which are not divisible; or, to put the point another way, there are points which have no point between them. If time is discrete, then there are indivisible instants; or, to put the point another way, there are pairs of times which are such that there is no time in between them.

Can you think of a mathematical series which, unlike the rational numbers, is discrete?

One can think of Zeno's strategy like this: he begins with the assumption space and time must be either continuous or discrete. He then proceeds to show that **either** assumption leads to the conclusion that motion is impossible.

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The Achilles

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Space and time are discrete

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It is useful to begin with the most well-known of Zeno's paradoxes: the Achilles.

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We assume two things about Achilles and the Tortoise.

First, we assume that Achilles always takes some amount of time to cover a given distance.

Second, we assume that the Tortoise, even though slow, is quite persistent; in particular, the Tortoise is in constant motion, so that the Tortoise covers some distance in every interval of time, no matter how small that interval of time.

Remember that we are assuming that space and time are infinitely divisible; so the amount of distance covered by the Tortoise in very small amounts of time can be arbitrarily small.

Tortoise’s head start

Now the race begins. Achilles sets off after the Tortoise.

Achilles eventually makes it to the point where the Tortoise started the race; but of course it takes him some finite amount of time to do so. Let’s call the this amount of time **t1**.

We know that the Tortoise, while slow, is persistent — so we know that the Tortoise has also moved some distance during the interval of time. Of course, he does not move as far as Achilles, but he does move.

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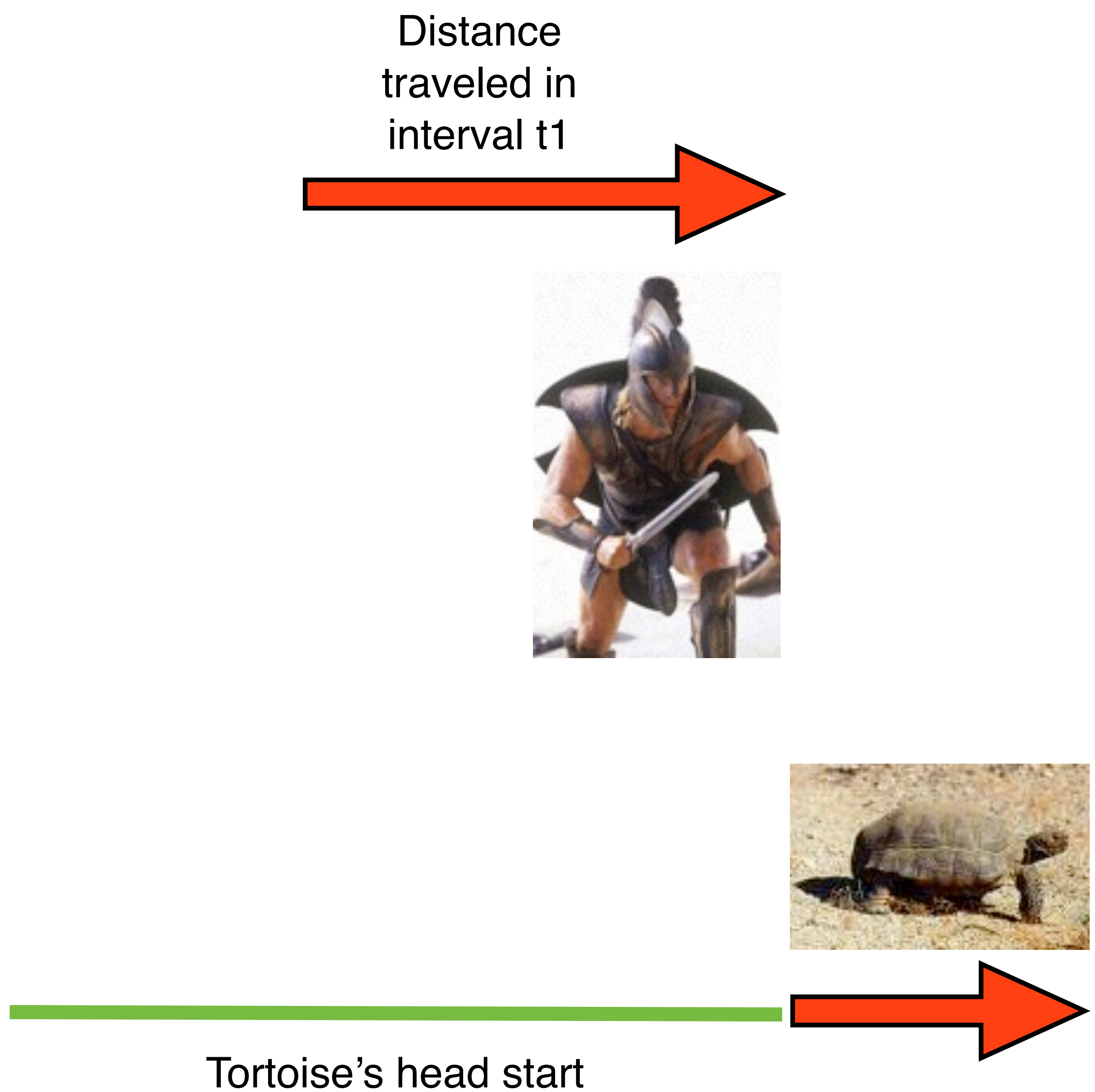
So, at the end of t1, the Tortoise's lead over Achilles has shrunk — but Achilles still has not caught him.

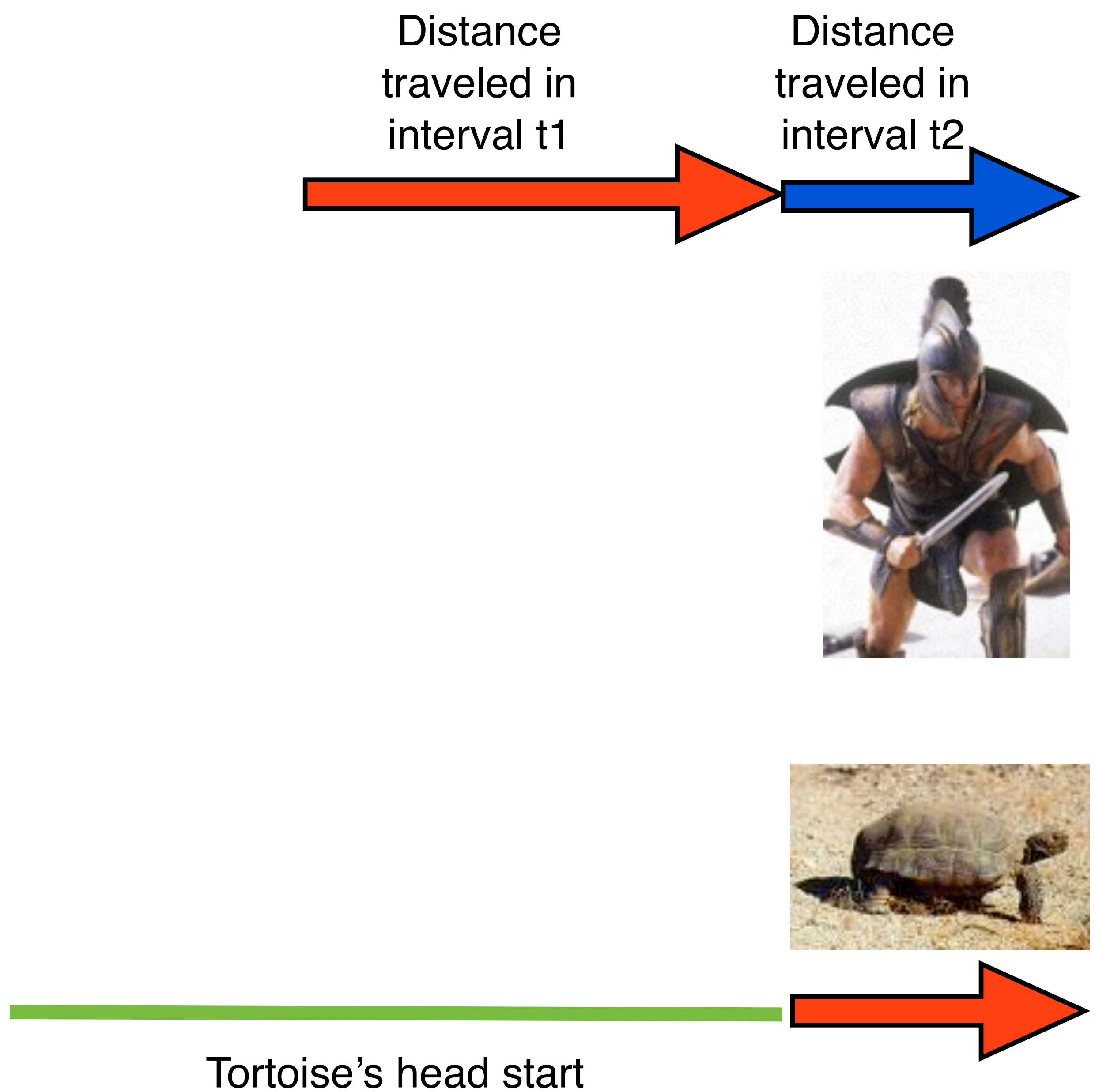
Achilles, of course, has not given up; he too is persistent, and faster than the Tortoise.



Achilles, of course, has not given up; he too is persistent, and faster than the Tortoise.

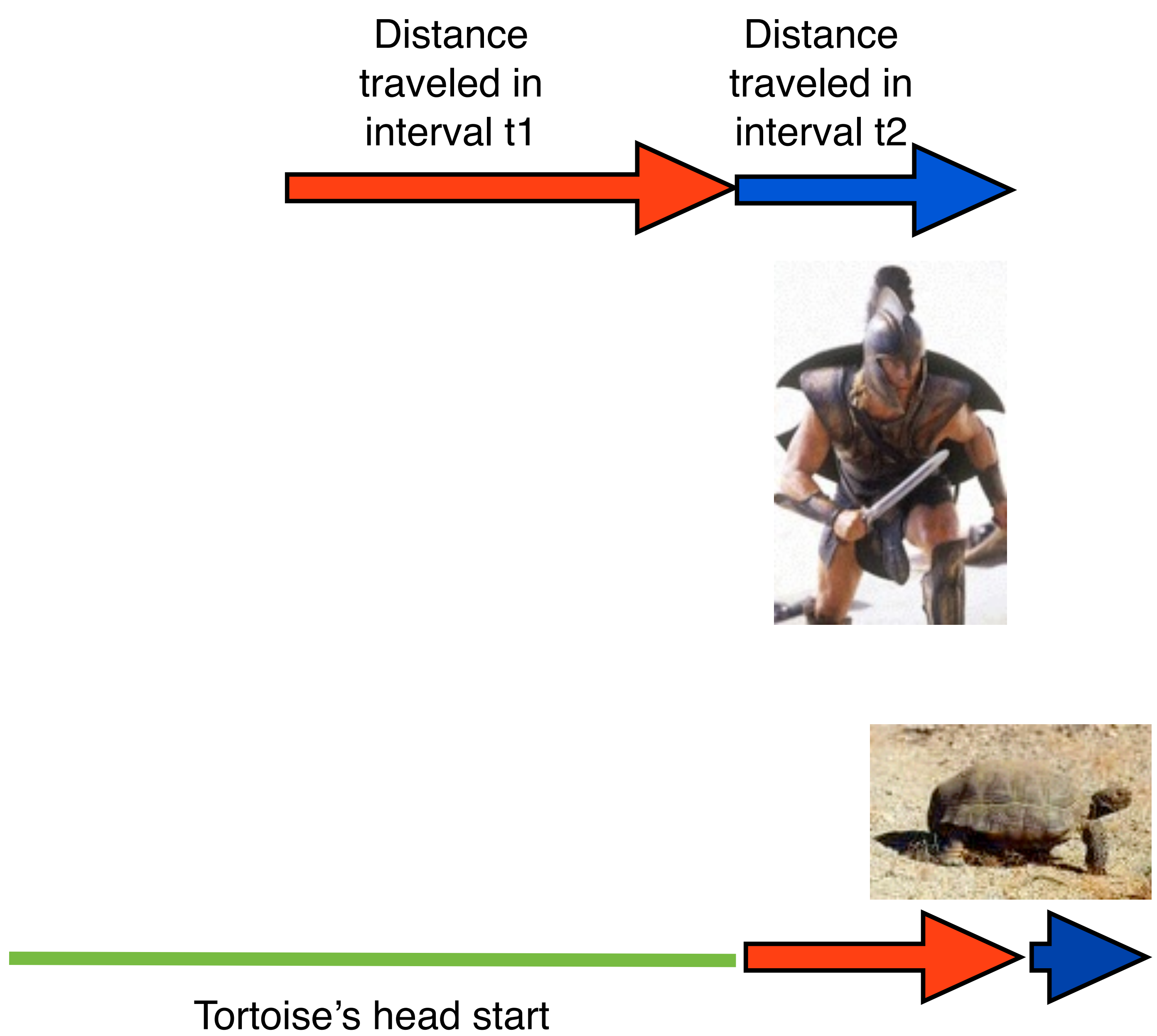
Pretty quickly, Achilles reaches the point that the Tortoise reached at the end of t1. But Achilles, while quite fast, is not infinitely fast; so this journey takes him a certain amount of time. Let's call this interval of time t2.

Has Achilles now caught the Tortoise, at the end of t2? Not quite. The Tortoise, after all, has moved some distance during t2 as well. Not, of course, as far as Achilles — but he has covered a little bit of ground.



So, at the end of t2, the Tortoise is still ahead of Achilles.

Suppose that we considered t3, t4, t5, and so on — would we ever get to an interval of time at the end of which Achilles had caught the Tortoise? It seems not. After all, it always takes Achilles some finite amount of time to catch the Tortoise, and during that finite amount of time, the Tortoise will always have covered some distance.



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But we know that this is absurd. Indeed, it seems very plausible that **if motion is possible at all**, it is possible for one thing to catch another thing from behind. But this seems to be what Zeno has shown to be impossible.

Of course, he has not **quite** shown that this is impossible — he has only show that this is impossible on the supposition that space and time are continuous. Why does Zeno's argument here depend on that assumption? How, in other words, could one respond to Zeno's argument if space and time were not continuous, but discrete?

Keeping the role played by this assumption in mind will help us to understand what's going on in the other paradox targeted at the assumption that space and time are continuous: the Racetrack.

Whereas the Achilles attempts to show that nothing can ever catch anything else from behind (so long as the former is moving at a finite speed and the latter never stops moving), the Racetrack attempts to show directly that it is impossible for anything to move any distance at all.

The idea behind the argument can be laid out informally as follows:

Imagine that you are trying to move from point A to point B. Suppose C is the midpoint of the distance from A to B. It seems that you have to first get from A to C, before you can get from A to B. Now suppose that D is the midpoint between A and C; just as above, it seems that you have to first get from A to D before you can get from A to C. Since space is infinitely divisible, this process can be continued indefinitely. So it seems that you need to complete an infinite series of journeys before you can travel any distance - even a very short one!

We can lay this out more carefully as an argument for the conclusion that it is impossible to move any finite distance in a finite time as follows:

1. Any distance is divisible into infinitely many smaller distances.
2. To move from a point x to a point y, one has to move through all the distances into which the distance from x to y is divisible.
3. To move from one point to another in a finite time, one has to traverse infinitely many distances in a finite time. (1,2)
4. It is impossible to traverse infinitely many distances in a finite time.

C. It is impossible to move from one point to another in a finite time. (3,4)

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The premises all seem plausible, the logic appears impeccable, but the conclusion is clearly false — what's going on here?

It is hard to reject premises 1 or 2, given our assumption that space and time are continuous. So attention focuses on premise 4: the assumption that it is impossible to traverse infinitely many distances in a finite time.

Why does premise 4 seem plausible? An initial thought is that premise 4 seems plausible because anyone who travels infinitely many finite distances will have to travel an infinite distance; and no one (at least, no one traveling at a finite speed) can do this in a finite time.

This argument is convincing if the following claim is true:

> The sum of any infinite collection of finite journeys is infinite.

But here we need to be a bit careful, as this claim has two different interpretations:

[A] Any finite distance is such that, covering that distance infinitely many times requires traveling an infinite distance.

[B] Taking infinitely many journeys, each of which covers some finite distance or other, requires traveling an infinite distance.

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Is there any reason to think that these differ in any important way?

To see the difference, it might be worth considering two different sorts of sums of infinite series. First, one corresponding to [A]:

$$\tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \ldots$$

But now consider the sum of an infinite series corresponding to [B], which does not require that each of the numbers in the series be the same:

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + 1/16 + \ldots$$

What is the sum of this series? Does this matter for our evaluation of premise (4) in the above argument? Does the Racetrack show that we must traverse a single finite distance infinitely many times, or just infinitely many distances?

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Let's pause for a moment to think about the connection between the Racetrack and the Achilles. If (4) is false, then we have solved the Racetrack. Have we also solved the Achilles?

Suppose we grant that one can travel infinitely many distances (each of which has some finite length) without traveling an infinite distance. Given this, is there any reason to think that one **can't** travel infinitely many distances in a finite time? In other words, is there any reason to think that premise (4) is true?

One might try to show that there is something incoherent in the idea that infinitely many events of a certain sort could take place in a finite time. This is the target of the example of "Thomson's lamp."

Thomson's lamp

A lamp is turned on and off an infinite number of times between 3:00 and 4:00 one afternoon. The infinite series of events then can be represented as follows:

on, off, on, off, on, off ....

and so on, without end. Because there is no end to the series, every "on" is followed by an "off", and every "off" is followed by an "on."

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Does it follow from our description of the lamp that at the end of the series, the lamp is neither on nor off?

Does it make sense to ask whether the lamp is on or off at **the first moment** after the end of the series?

Does it make sense to ask whether the lamp is on or off at 4:01, given the stipulation that, after the series, it is never turned on or off?

If we agree that Thomson's lamp is impossible, then that might make you think that completing an infinite series of tasks in any amount of time is impossible, which in turn might make you think that premise (4) is true.

If premise 4 is true, then it looks like the Racetrack (as well as the Achilles) is a pretty strong argument against the possibility of motion **given the supposition that space and time are continuous**. So let's turn to the other possibility: the possibility that space and time are discrete.

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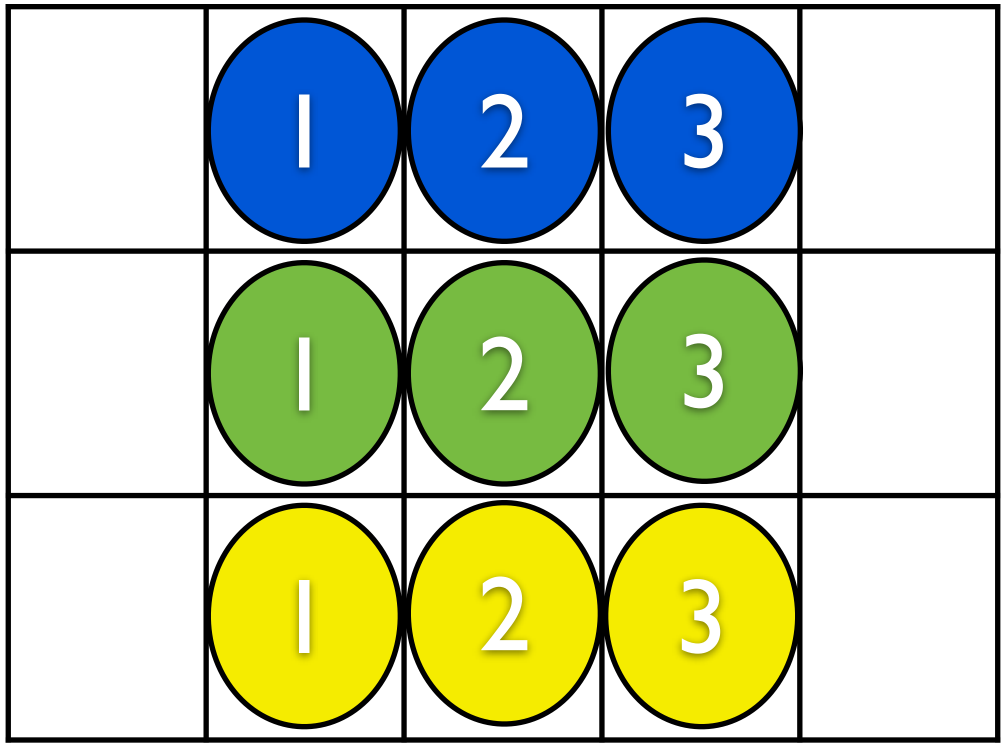
Let's begin with the Stadium - an argument which Sainsbury does not discuss.

We are now assuming that space and time are discrete, which means that there can be points in space which are genuinely adjacent, in the sense that there are no points in between them. Suppose that the following is a grid of such adjacent points.

Now suppose that we occupy these points with certain particles, as follows:



Let's call the time at which the particles are thus arranged Time 1.

Now let's suppose that the blue particles are all about to move one space to the left, and the yellow particles are all about to move one space to the right.

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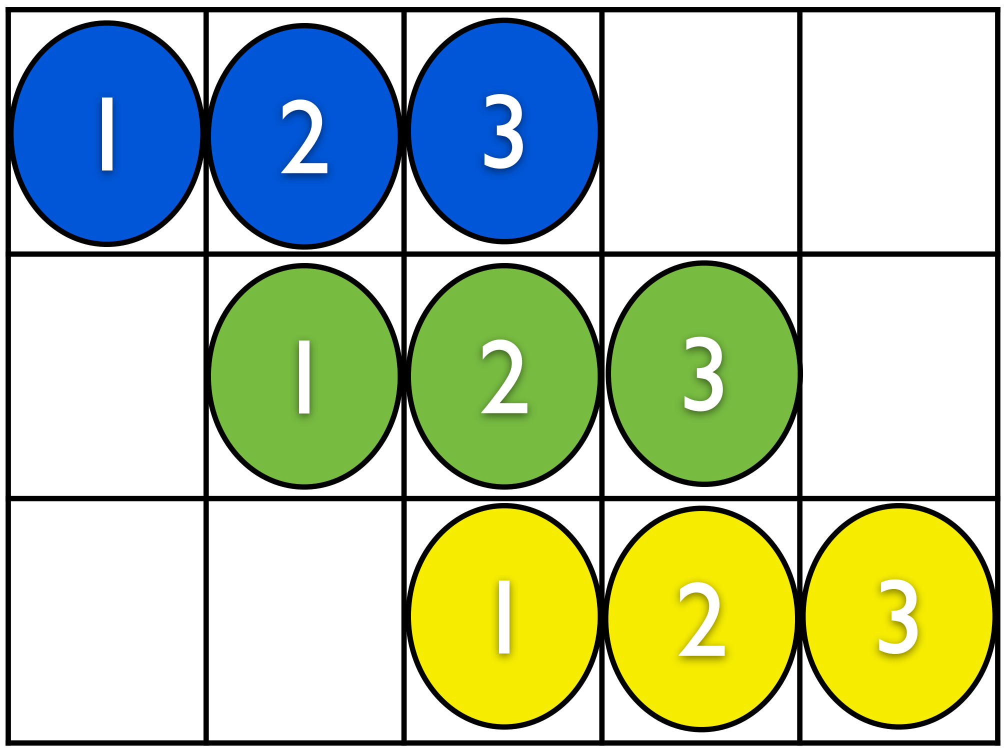

Let's call the time after this movement is complete Time 2.

We are supposing that space and time are discrete, so we can assume that Time 1 and Time 2 are adjacent times, in the sense that there is no time between the two — just as there is no space between the boxes in our chart.

If you think about it, something very odd happened between Times 1 & 2.

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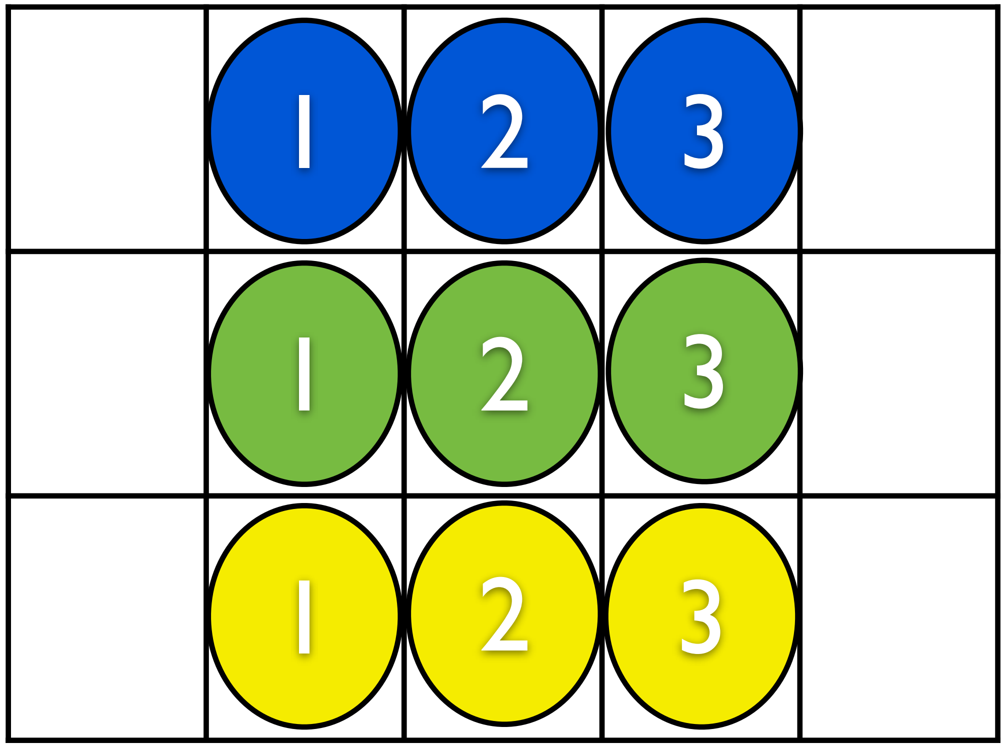

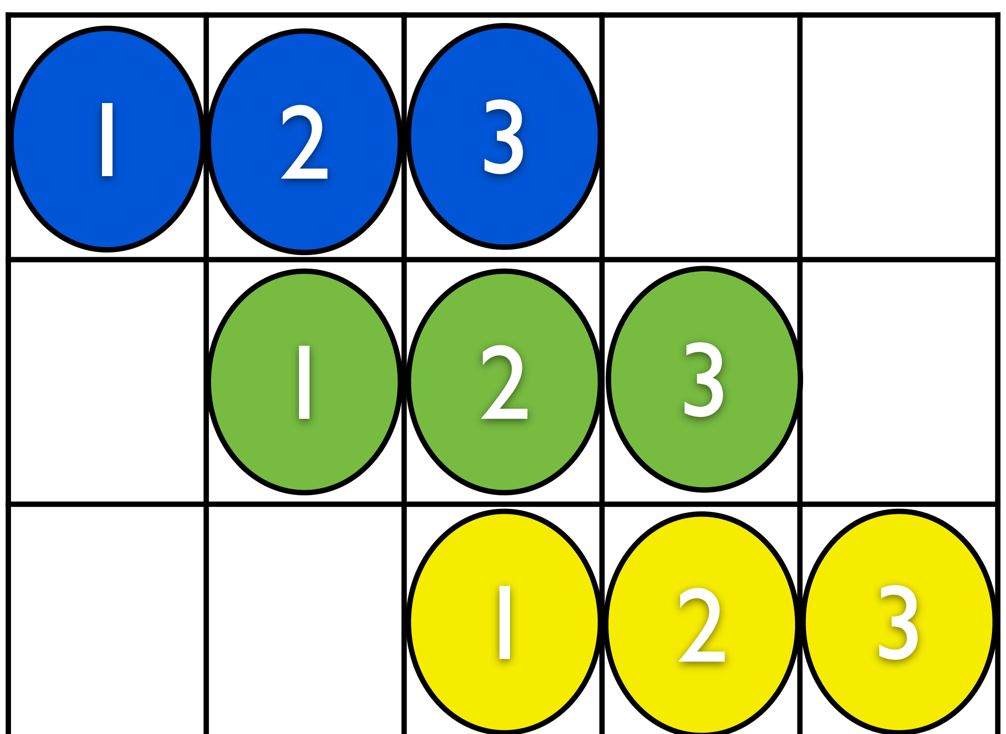

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Look at, for example, Yellow-2 and Blue-3. At Time 1, Yellow-2 is to the left of Blue-3. At Time 2, Yellow-2 is to the right of Blue-3. **But the two never passed each other**. After all, they did not pass each other at Time 1, and did not pass each other at Time 2, **and there was no time in between**.

But this seems impossible. It seems impossible for objects to switch left-right orientations without at some point being "even" with each other. But if this really is impossible, then it seems to follow that motion is impossible if space and time are discrete.

Is this a convincing argument?

Zeno's final paradox is called "The Arrow."

Consider an arrow shot from a bow, and imagine that space and time are discrete.

Consider an indivisible moment in time. Does the arrow move **during** that instant? It seems that it cannot since, if it did, the instant would be divisible — the arrow would have to be in one place for one part of the instant, and in another part for another. But if instants have parts, then they are not indivisible.

Can it move **between** instants? No, because there are no times between instants.

But if it cannot move during instants, and cannot move between them, it cannot move. So motion is impossible.

This argument can be laid out as follows:

1. During any one instant, an arrow does not move.
2. Nothing happens between one instant and the next.
3. The arrow does not move between instants. (2)

C. The arrow does not move. (1,3)

Is this argument valid? Could things move, even if they do not move either at individual instants or between them?

One might say that motion is something that neither happens at instants nor between them. Rather, motion is just a matter of being in one place at one time, and another place at the next time. Real motion then becomes a bit like the motion in film movies, which is just a matter of projected objects being in one location on one frame of the film, and another on the next.

Could this really be all that there is to motion? Consider a billiard ball in motion over some spot X on the pool table at time t, and another ball just sitting on spot X on an identical pool table at that time. Isn't it weird to think that there is no difference between those balls at that time?