The Russell-Myhill paradox

Jeff Speaks
PHIL 93503

November 27, 2023

Our main topic is the Russell-Myhill paradox (sometimes also called 'Russell's paradox of propositions'). But to understand that paradox it helps to have a grip on two bits of set theory: Russell's paradox of sets (usually just called 'Russell's paradox') and Cantor's theorem.

1 Russell's paradox

Russell's paradox arises from Basic Law V of Frege's logicist system. That law can be stated using higher-order quantification like this:

$$\forall F \forall G \ (\text{extension}(F){=}\text{extension}(G) \leftrightarrow \forall x(Fx \leftrightarrow Gx))$$

where 'extension' combines with a predicate to deliver the set of things to which the predicate applies. This was part of Frege's definition of numbers as sets of equinumerous sets.

Remember that last time we were using lambda notation to pick out the sorts of things that are the values of higher order variables like 'F' ands 'G' in the above. We might then form expressions like 'extension($\lambda x.x$ is red)' — the set of red things.

Russell's paradox arises when we consider 'extension($\lambda x.x \notin x$)' — the 'Russell set' R of all the non-self-membered things. Both $R \in R$ and $R \notin R$ lead to contradiction.

In more modern notation a principle that would give rise to Russell's paradox would be the following unrestricted comprehension axiom:

$$\forall F \exists S \forall x \ (x \in S \leftrightarrow Fx)$$

Or, without predicate quantification: the claim that every instance of the corresponding schema is true.

One way of thinking of sets: the collection of things satisfying some predicate. On this conception, the falsity of the unrestricted comprehension principle looks puzzling.

An alternative way of thinking about sets: the 'iterative conception' of sets, in which the universe of sets is 'built up' step by step.

One can then ask: if sets were things which are built up in this way, what axioms would govern the universe of sets? A descendant of unrestricted comprehension survives as the axiom of separation, which says that for any set S and condition F there is a subset S containing only the elements of S which satisfy F.

In standard set theory the sets are all 'pure sets;' there are no sets of dogs, cats, etc.

Of course, when we are doing semantics (and lots of other things) we want sets which have things other than sets as members. Suppose that we want to expand our theory to include sets of dogs, cats, and other 'ur-elements.' How do we do this in way which ensures that we will have enough sets to do all of the work we want done? One natural thought: add an axiom saying that for any condition on ur-elements, there is a set containing just the ur-elements which satisfy that condition. What could go wrong? After all, cats and dogs can't be elements of themselves, etc.

2 Cantor's theorem

One of the axioms of standard set theory is the power set axiom. This says that for every set S, there is a set $P(S)$ — S's 'power set' — whose elements are all of the subsets of S. Cantor's theorem says that the power set of any set is larger than (has greater cardinality than) the set.

Obvious for finite sets. How about infinite sets?

Suppose for reduction that A and $P(A)$ are the same size. Then there is some one-to-one mapping f from elements of A to elements of $P(A)$.

Now consider the set B defined as follows: $\{x : x \in A \,\&\, x \notin f(x)\}$. So something is a member of B just in case it is in A but not in the element of B to which it is mapped by f.

B is a subset of A. So it is an element of $P(A)$. So, if f is a one to one mapping from A to $P(A)$, there must be some element x of A such that $f(x) = B$.

So suppose for reductio that a is such an element, and that $f(a) = B$. Now ask: is $a \in B$?

Suppose that $a \in B$. B is defined to include only the elements of A which are not elements of the things to which they are mapped by f. Since a is mapped to B by f, $a \notin B$.

Suppose instead that $a \notin B$. B is defined to include every element of A which is not an element of the things to which they are mapped by f. Since a is mapped to B by f, $a \in B$.

So there is no element of A which f maps to B. Since B is an element of $P(A)$, there is no one to one mapping from A to its power set. So A's power set is larger than A.

3 The Russell-Myhill paradox

The Russell-Myhill paradox was first presented in an appendix of Russell's *Principles of Mathematics* (1903):

> 'If m be a class of propositions, the proposition "every m is true" may or may not be itself an m. But there is a one-one relation of this proposition to m: if n be different from m, "every n is true" is not the same proposition as "every m is true". Consider now the whole class of propositions of the form "every m is true", and having the property of not being members of their respective m's. Let this class be w, and let p be the proposition "every w is true". If p is a w, it must possess the defining property of w; but this property demands that p should not be a w. On the other hand, if p be not a w, then p does possess the defining property of w, and therefore is a w. Thus the contradiction appears unavoidable.'

Among the propositions, there will be some which are, for some set of propositions, the proposition that all propositions in that set are true. Let's call these the 'set describing' propositions. Some propositions will be members of the set they describe and some will not. Examples:

> Every proposition about cats is true.
>
> Every proposition expressed by a sentence of *Naming and Necessity* is true.
>
> Every quantified proposition is true.
>
> Every atomic proposition is true.

Let w be the set of all and only the set describing propositions which are not members of the set they describe: $\{p : \exists S\ (p \text{ describes } S\ \&\ p \notin S)\}$.

We can now define the proposition p as the proposition that every member of w is true. p is then a set describing proposition, and w is the proposition it describes.

Suppose first that $p \in w$. Then it is a set describing proposition that is in the set it describes; so $p \notin w$.

Suppose instead that $p \notin w$. Then it is a set describing proposition which is not an element of the set it describes; so $p \in w$. Uh oh.

We can represent the assumptions which generate the contradiction as follows (Uzquiano (2015)):

(1) If m is a set of propositions, there is a proposition *every proposition in m is true.*

(2) If m and n are distinct sets of propositions, then the proposition *every proposition in m is true* is distinct from the proposition *every proposition in n is true.*

(3) There is a set w of all propositions of the form *every proposition in m is true*, for some set of propositions m to which the proposition does not belong.

Why is assumption (2) essential to deriving the contradiction? Our definition of the set w would still make sense without this assumption, right? What if we switched the quantifier in the definition of w to $\forall$?

Another way to get a contradiction is to add to (1) and (2) the assumption

(4) There is a set of all propositions.

Call this set S. Then the power set axiom implies that there is a set $P(S)$ of all subsets of the set of propositions. Cantor's theorem implies that there is no mapping from the members of S to the members of $P(S)$. But every member of $P(S)$ will be a set of propositions. If (1) and (2) are true there will be a one-to-one mapping from the set describing propositions (all of which are elements of S) to the elements of $P(S)$.

3.1 Coarse-grained theories

Which of the theories of propositions we've discussed would seem to have a built-in response to the paradox?

Some see this paradox as a strong argument for coarse-grained views of propositions.

For just the same reason, some see it as a strong argument against 'structured' theories of propositions. But it is less an argument against theories which think of propositions as having constituents in some metaphysically deep sense than it is an argument against theories which think of propositions as suitably fine-grained.

What theses about the fine-grainedness of propositions would seem to be open to the paradox? One popular candidate: if p is a monadic predication of some property F of x and q is a monadic predication of G of y, then if $p = q$ then $F = G$ and $x = y$. Or, better, some more general principle of this kind. That would seem to deliver the truth of (2). For ease of reference call views which accept principles like this 'structured.'

Suppose for now that the paradox rules out structured views. Exactly how coarse-grained must propositions be to avoid the argument? Possible worlds views get around it; but there are plenty of views in between that and structured views.

The question is whether any of the in-between options are principled. For example, one could endorse a limited structured view which only applied to predications of certain types. But why hold the fine-grained view for some predications but not others?

Another possible in between view: the view that logically equivalent propositions are identical. Follow up: what is logical equivalence for propositions? One answer involves going for the sort of higher-order language we talked about last time. Then p, q will be logically equivalent just in case it is a truth of the relevant system of higher order logic that $p =_t q$. Is there strong reason to prefer this sort of view to the possible worlds/intensionalist view?

3.2 Structured theories

How might the fan of structure reply?

Consider the view that propositions are sets of possible + impossible worlds. As we have seen, that can deliver a very fine-grained view of propositions. Does the fact that this view identifies propositions with sets interact with the paradoxical reasoning in any interesting way?

Kment (2022) tries to give a principled response to the paradox in terms of pre-theoretically attractive principles about grounding. The key assumptions, roughly:

- Grounding is an explanatory non-causal transitive (simplifying a bit here) relation.
- No fact grounds itself.
- A set's existence is grounded in the existence of its members.
- A proposition's existence is grounded in the existence of its constituents.

We might then think of entities like propositions and sets in a way analogous to the iterative view of sets.

If these principles are true, then it looks like one of (1) and (3) must be false. Consider the set w. This is defined in terms of the set of set-describing propositions. In order to generate the paradox, the problematic proposition p must (i) be in this set, and must also (ii) be about w. But both cannot be true. If (i) is true, then w is grounded in p; if (ii) is true, then p is grounded in w.

(3) is arguably the one to reject. If a set exists, how could the the corresponding set describing proposition fail to exist?

We can still construct, at any level n of the hierarchy, the set S_n of set-describing propositions at levels $< n$ that are not members of the relevant set. And we can then construct a proposition that says that every member of S_n is true. But no paradox from saying that the proposition is not an element of the set.

Recall that even if (3) is false we still get contradiction from (1), (2), and (4). Reply: deny that there is a set of all propositions.

Last time we talked about higher-order languages. Proponents of theorizing in these languages point out that plausible higher-order translations of the structured view are inconsistent in some natural higher-order logics. (See e.g. Dorr et al. (2021).)

References

Cian Dorr, John Hawthorne, and Juhani Yli-Vakkuri, 2021. *The Bounds of Possibility: Puzzles of Modal Variation*. Oxford: Oxford University Press.

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